Modern Perspectives on Einstein's General Theoryof Relativity

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Abstract: This paper addresses the mathematics of Einstein's General Theory of Relativity with regard to the force of gravity. A unique approach is used, with some historical background, to show the progression of events that have led to our current understanding of how energy, mass, and light are interrelated. Our viewpoint includes the work of a physical chemist, an organic chemist, and a biologist, interested in how the universe works. The mathematics described for Einstein's General Theory of Relativity in this paper incorporates use of Tensor calculus, which involves a set of rules and methods for mathematical objects that have an arbitrary, but known, number of situations. This approach focuses on how mathematics can be applied, quantitatively, to explain Einstein's General Theory of Relativity for gravity; such as that used to calculate the slight procession of the ellipticalorbit of the planet Mercury about the sun every hundred years; the slight bending of starlight by the sun; and the slight time dilation of Global Positioning System satellites. Gravity is the force one feels at the surface of the earth and matches that which one would observe if they are inside a rocket that is accelerating at a rate of 9.80 meters per second-squared (9.80 m/s^2) or 32.2 feet per second-squared (32.2 ft/s^2) in outer space. In this paper, we modified Newtonian theory by using the Schwarzschild Metric to derive kinetic and negative gravitational energy by guiding the reader through the mathematics with key references.

Introduction

We are professors, fascinated by Einstein's Theory, and the equation, $E = mc^2$, which for all intents and purposes, explains much of the how and why of the basic nature of light and gravity exists. As humanity continues to explore beyond our world, we are convinced that Einstein's Theory will hold in most cases, but anxious to see how discoveries may show deviations in what we currently understand. What has been interesting is that Einstein's General Theory of Relativity has applications in other areas of science, such as electromagnetism, global navigation satellite systems, nuclear power, and even the very existence of life itself.

Recently, a photograph of a black hole was generated (The Event Horizon Telescope Collaboration, 2019). Many do not appreciate that the initial conjecture of black holes was suggested by Karl Schwarzschild (Schwarzschild, 1916; see translation Schwarzschild, 2003), by solving the Einstein Field Equations a few months after Albert Einstein published his Theory on General Relativity (Einstein, 1915a; 1915b; 1915c; 1915d; see translations "Collected Papers of Albert Einstein" in 1987). There were other investigators, including Johannes Droste (1916) and David Hilbert (1917) who did much of the work to solve the Einstein Field Equations and the conjecture of black holes. It was Schwarzschild who attempted to solve the Field Equations for the static case (Schwarzschild, 1916; see translation Schwarzschild, 2003) a few months after Albert Einstein published his Theory on General Relativity (Einstein, 1915a; 1915b; 1915c; 1915d; see translations "Collected Papers of Albert Einstein" in 1987). Droste and

Hilbert were able to take Schwarzschild's work further and derived what some refer to as the "Schwarzschild Metric." Our tactic describes the Schwarzschild Metric derivation to solve the Einstein Field Equations in the vacuum solution of solving for the Ricci Tensor (Ricci and Levi-Civita, 1901; see translation, Ricci-Curbastro, 1975) and the Ricci Scalar (Ricci, 1904) by considering the mass density in the Einstein Field Equations to be equal to zero. The Cosmological term is also assumed to be zero. Schwarzschild and Hilbert considered a body to be stationary, which is a non-rotating body that can approximate real life cases such as the sun and earth. Hilbert first derived a solution that is called in modern parlance, an "Event Horizon" or what Hilbert called the "Schwarzschild Radius." If such a body has a Schwarzschild Radius that is larger than the radius of the object, the outcome is what is called, in modern language, a black hole. In the area around a black hole, a light-like vector that is stationary along the space axis may be produced. This is considered to be the Event Horizon of a black hole in which light itself cannot escape. It is possible to calculate the bending of light around a massive object in this scenario. Einstein calculated the bending of light and confirmation of this was provided in an expedition during a total solar eclipse. Hence, Einstein developed a new way of describing the nature of light that many believed was needed and a new theory of gravity emerged (Coles, 2019).

At the end of the 19th century, there was a need for new thinking on the nature of light and its propagation. Light was considered to be a wave and therefore needed some media to propagate in space. The concept of "luminiferous aether" (e.g. ether) was used as the sole propagating medium, and it should be possible to measure the speed of light as light passes from the sun with and against the movement of the ether. The motion of the ether should add or subtract as vector quantities from the velocity vector of the light. The speed of light experiment by Michelson and Morley (1887) shows no such change in the speed of light.

This lack of need for the ether as evidenced by this experiment, and by the Maxwell Equations of Electromagnetism (Maxwell, 1861a; 1861b; 1862a; 1862b; see Fleisch, 2010), which are independent of motion showing a constant velocity of light, also had a profound change on another paradigm. The Newtonian concept (Newton, 1687; see translation Newton, 1999) of absolute space in which a position can be determined with coordinates with the ether with a fixed position was overturned. Minkowski (1908; see translation Minkowski, 2012) developed the concept of space-time in which the geometry of space and time for any object exists with three dimensions of geometry and one dimension of time.

Although Einstein never personally refuted the presence of the ether, this lack of proof of its existence allowed Einstein to develop his General Theory of Relativity from the work of Minkowski. In this theory, relative space means all inertial frames can be considered to be equally valid in that all the laws of physics are the same within each frame. In having a fixed speed of light and all frames being equally valid, time and space might appear to be either compressed or time appears to run more slowly to an observer in another inertial frame of reference that would be very noticeable as one approaches the speed of light. An important consequence of this was a new insight concerning gravity. Because space-time sets up a manifold, one can think of it as a trampoline as space-time with a heavy object as a star represented by a bowling ball. One could try to roll the lighter object, such as a marble around the heavy object, but the depression in the trampoline would cause the marble to appear to swing around the heavier object. Gravity was the result of this disturbance of space-time.

Schwarzschild, Droste, and Hilbert used differential geometry to solve the Einstein Field Equations in the first non-trivial solution. Johannes Droste was the first to use the final Einstein paper to develop a non- trivial solution that shows repulsive gravity (Droste, 1916) Some of the mathematics used by Einstein and Schwarzschild to solve this equation was developed before Luigi Bianchi developed the geometry of topology (Bianchi, 1891) and fellow Italian, Gregorio Ricci (also referred to as Gregorio Ricci-Curbastro) developed the use of Tensor calculus that describes such a space. Tensors are geometric objects that map for the point of physics, two vectors to a new tensor. The calculus of these differential shapes can be treated by field equations that the first investigators of relativity (Schwarzschild, Droste, and Hilbert) used. In the following description, a set of vectors in such a geometric space is called a basis. Such a basis is described by a multidimensional array. In Tensor calculus, when a vector is invariant or whenever the direction and magnitude of the resulting vector is the same, its components of the field equation must transform to a contravariant rule to keep the direction and magnitude of the resulting vectors the same. In differential geometry, the Christoffel symbols (Γ) that are used in the field equations (Christoffel, 1869; see Eisenhart, 1940) are the array of numbers that describe a metric connection which is the topology, or surface geometry, that can be described by a vector bundle with a metric bundle.

Einstein's Special Theory of Relativity and Minkowski Space in Rectangular Cartesian Coordinates

Much of the focus on Einstein's work revolves around five papers published in 1905 and 1906 (Einstein, 1905a; 1905b; 1905c; 1905d; 1906). Einstein's Special Theory of Relativity states that the speed of light is a universal constant in an inertial reference frame, being that the sum of all forces is equal to zero for an object at rest or is in linear motion at a constant speed. From this result, Albert Einstein developed the theory that energy E equals the relative mass m times the speed of light squared c^2, where c represents the speed of light (Einstein, 1905d):

$$E = mc^2 = \left(\frac{m_0}{\sqrt{1 - v^2/c^2}}\right)c^2, \quad (1)$$

and the relativistic mass m is the function of object's rest mass m_0, the object's speed v, and the speed of light c as shown in Equation 1.

Another interesting effect with Einstein's Special Theory of Relativity is time dilation. If one travels near the speed of light from our solar system to one of the nearby stars, their biological clock will move at a slower pace than those remaining on planet earth. This is referred to as Lorentz time dilation (Lorentz, 1899) whereby proper time change for one in the moving space craft is referred to as $\Delta\tau$ and time change for one on planet earth is Δt. The change in proper time $\Delta\tau$ for the fast-moving space craft will be equal to the following expression as a function of time Δt on planet earth, the speed v of the moving space craft, and the speed of light c:

$$\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}. \quad (2)$$

Thus, one has the following derivative function $dt/d\tau$, the derivative of earth time t with respect to time τ, for the individual flying away in the fast moving space craft:

$$\frac{dt}{d\tau} = \frac{\Delta t}{\Delta\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (3)$$

In other words, a person moving near the speed of light will age at a slower rate than a person who is at rest with our solar system. If one squares Equation 3, takes the reciprocal, and then multiplies through with the speed of light squared c^2, the following important expression is derived:

$$c^2 - v^2 = c^2\left(\frac{d\tau}{dt}\right)^2 = c^2\frac{d\tau^2}{dt^2}. \tag{4}$$

Then we can multiply dt^2 through both sides of Equation 4 to obtain

$$c^2dt^2 - v^2dt^2 = c^2d\tau^2. \tag{5}$$

Realizing that the quantity v^2 is the dot product of the velocity vector $\mathbf{v}$ with itself such that

$$\mathbf{v}\cdot\mathbf{v} = v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}} + v_z^2\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = v_x^2 + v_y^2 + v_z^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2. \tag{6}$$

Equation 5 becomes the following differential expression since dt^2 divides out to one for v^2dt^2:

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2d\tau^2. \tag{7}$$

After multiplying Equation 7 by negative one, –1, the next expression is the linear function for what is referred to as Minkowski space or space-time in the four dimensions of x, y, z, and time t, with regards to proper time τ:

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = -c^2d\tau^2. \tag{8}$$

Note that Equation 8 is simply the dot product of the any position vector in Minkowski space or space-time as referred to in Einstein's Special Theory of Relativity with the three unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ as in Cartesian rectangular coordinates. Plus, there is an additional time unit vector $\hat{\mathbf{t}}$ for the fourth dimension of time t and unit vector $\hat{\boldsymbol{\tau}}$ for proper time τ concerning distance traveled in Minkowski space at the speed of light c:

$$ic\,d\tau\,\hat{\boldsymbol{\tau}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} + ic\,dt\,\hat{\mathbf{t}}. \tag{9}$$

In Equation 9, $ic\,d\tau\,\hat{\boldsymbol{\tau}}$ represents velocity vector of an object moving through the four dimensions of Minkowski space, and the square of the distance traveled in Minkowski space is $-c^2d\tau^2$. Thus, all objects move at the speed of light in the four-dimensional Minkowski space. Two objects appearing to be at rest with one another are moving at the speed of light along the t-axis relative to one another.

Concerning the derivation of Equation 1, the magnitude of the momentum vector is the relativistic mass m times the speed v of a moving object:

$$p = mv = \left(\frac{m_0}{\sqrt{1 - v^2/c^2}}\right) v. \tag{10}$$

By Newton's Second Law of Motion, force is equal to derivative of momentum with respect to time, so by using Equation 10 for the relativistic momentum, force is the following definition by Newton's Second Law in one-dimension for linear motion:

$$F = \frac{dp}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}. \tag{11}$$

Since by definition, work or change in kinetic energy is equal to the integral of force times distance, and when including relativistic momentum, work or kinetic energy is the following integral for an object accelerated by a constant force from rest to final velocity value v:

$$W = KE = \int_0^v \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}dx = \int_0^v \frac{m_0}{(1 - v^2/c^2)^{3/2}}\, v\, dv = mc^2 - m_0c^2. \tag{12}$$

It is important to note that the integral in Equation 12 is the derivation for Albert Einstein's famous equation relating mass and energy given in Equation 1, $E = mc^2 = KE + m_0c^2$. With regard to the expression in Equation 12, for speeds much less than the speed of light $(v \ll c)$, the kinetic energy for an object with rest mass m_0 is nearly equal to the classical kinetic energy value $KE = \frac{1}{2}m_0v^2$ using the binomial expansion for $1/\sqrt{1 - v^2/c^2} \approx 1 + \frac{1}{2}v^2/c^2$ for $v \ll c$. Note that Equation 12 applies to flat or Minkowski space that is not stretched. Stretched Minkowski space is how Einstein's General Theory of Relativity explains the force of gravity.

Spherical Polar Coordinates

Usually one uses the Cartesian coordinate system for three-dimensions comprised of x-, y-, and z-axes. The Cartesian coordinate system is appropriate when one involves the calculation of linear motion. However, when one wants to mathematically describe curved motion, it is much more conveniently to apply what is referred to as spherical polar coordinates r, θ, and ϕ. In vector notation, any position vector **r** in three-dimensions is represented below using Cartesian rectangular coordinates with orthonormal unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$:

$$\mathbf{r} = r_x\hat{\mathbf{x}} + r_y\hat{\mathbf{y}} + r_z\hat{\mathbf{z}}. \tag{13}$$

In addition concerning Equation 13 above, r_x, r_y, and r_z represents the x-, y-, and z-components of the position vector **r**, such that:

$$r_x = r\cos\phi\sin\theta; \qquad r_y = r\sin\phi\sin\theta; \qquad r_z = r\cos\theta. \tag{14}$$

Also, in Equations 14, r represents the magnitude of the position vector **r**:

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}. \quad (15)$$

Angle ϕ represents the direction of the position vector **r** about the x-axis in the xy-plane at $z = 0$, and angle θ represents the direction of the position vector **r** about the z-axis. Henceforth, the position vector **r** can be represented as the following function of the vector magnitude r and angles ϕ and θ as

$$\mathbf{r} = r\,\cos\phi\,\sin\theta\,\hat{\mathbf{x}} + r\sin\phi\sin\theta\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} = r\hat{\mathbf{r}}. \quad (16)$$

Normally in spherical polar coordinates, the unit vector $\hat{\mathbf{r}}$ is presented as the following function of angles ϕ and θ and orthonormal unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$:

$$\hat{\mathbf{r}} = \cos\phi\,\sin\theta\,\hat{\mathbf{x}} + \sin\phi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}. \quad (17)$$

In the Cartesian rectangular coordinate system, the orthonormal unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ have fixed directions and are always parallel with their corresponding x-, y-, and z-axes. On the other hand, in spherical polar coordinate system, orthonormal unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ do not have fixed directions and change their directions with a moving position vector **r** while remaining perpendicular or orthogonal to each other, with unit vector $\hat{\mathbf{r}}$ always being parallel with position vector **r**.

By definition for the tangent of any curved function $y = f(x)$, the slope of the tangent at x is equal to the derivative of the function $y = f(x)$ with respect to x, slope = $df(x)/dx$. Thus, unit vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the following derivative functions of $\hat{\mathbf{r}}$ with respect to θ and ϕ:

$$\hat{\boldsymbol{\theta}} = \frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \cos\phi\,\cos\theta\,\hat{\mathbf{x}} + \sin\phi\,\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}; \quad (18)$$

$$\hat{\boldsymbol{\phi}} = \frac{1}{\sin\theta}\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \frac{1}{\sin\theta}\left(-\sin\phi\,\sin\theta\,\hat{\mathbf{x}} + \cos\phi\,\sin\theta\,\hat{\mathbf{y}}\right) = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \quad (19)$$

For Equation 19 to be a unit vector such that the dot product of $\hat{\boldsymbol{\phi}}$ with itself is equal to one ($\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$), it is necessary to divide by the quantity $\sin\theta$.

Before discussing the mathematics of Einstein's General Theory of Relativity, it also necessary to give a discussion about classical physics of bodies in motion using spherical polar coordinates. The next section not only discusses the employment of spherical polar coordinates for classical physics, but it also includes some digression on Tensor calculus, the mathematics used in Einstein's General Theory of Relativity.

Velocity and Acceleration in Spherical Polar Coordinates using Tensor Calculus

With regard to the velocity vector **v** and the acceleration vector **a** in rectangular Cartesian coordinates, they are simply the first and second derivatives with respect to time t of the position vector **r** for linear motion:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr_x}{dt}\hat{\mathbf{x}} + \frac{dr_y}{dt}\hat{\mathbf{y}} + \frac{dr_z}{dt}\hat{\mathbf{z}}; \quad (20)$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2 r_x}{dt^2}\hat{\mathbf{x}} + \frac{d^2 r_y}{dt^2}\hat{\mathbf{y}} + \frac{d^2 r_z}{dt^2}\hat{\mathbf{z}}. \tag{21}$$

However, for curved motion with change in values of angles ϕ and θ as well as for the vector magnitude r with time, the mathematics becomes more complex which can be handled mathematically using Tensor calculus.

In spherical polar coordinates, the velocity vector $\mathbf{v}$ becomes the following first derivative of position vector $\mathbf{r}$ with respect to time:

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial \mathbf{r}}{\partial r}\frac{dr}{dt} + \frac{\partial \mathbf{r}}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \mathbf{r}}{\partial \phi}\frac{d\phi}{dt}, \tag{22}$$

and now one must use the chain-rule when taking the first derivative of $\mathbf{v}$ with respect to time t, since r, θ, and ϕ are all functions of time which includes the changing directions of the unit vectors $\hat{\mathbf{r}}$, $\widehat{\boldsymbol{\theta}}$, and $\widehat{\boldsymbol{\phi}}$ with curved motion. The first derivative of vector $\mathbf{r}$ with respect to r can be represented as $\mathbf{r_r}$ in Tensor calculus, as shown below, which is equal to the unit vector $\hat{\mathbf{r}}$ when taking the derivative of Equation 16 with respect to r:

$$\mathbf{r_r} = \frac{\partial \mathbf{r}}{\partial r} = \cos\phi\sin\theta\,\hat{\mathbf{x}} + \sin\phi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} = \hat{\mathbf{r}}. \tag{23}$$

Likewise, taking the derivative of vector $\mathbf{r}$ with respect to angle θ in Equation 16 results in the following expression defined as vector $\mathbf{r}_{\boldsymbol{\theta}}$, which ends up being equal to the product of the magnitude of the position vector $\mathbf{r}$ times the unit vector $\widehat{\boldsymbol{\theta}}$:

$$\mathbf{r}_{\boldsymbol{\theta}} = \frac{\partial \mathbf{r}}{\partial \theta} = r\,[\cos\phi\cos\theta\,\hat{\mathbf{x}} + \sin\phi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}] = r\widehat{\boldsymbol{\theta}}. \tag{24}$$

Then when taking the derivative of Equation 16 with respect to angle ϕ, this results in the next expression for vector $\mathbf{r}_{\boldsymbol{\phi}}$ which is equal to the quantity $r\sin\theta\,\widehat{\boldsymbol{\phi}}$:

$$\mathbf{r}_{\boldsymbol{\phi}} = \frac{\partial \mathbf{r}}{\partial \phi} = r\,[-\sin\phi\sin\theta\,\hat{\mathbf{x}} + \cos\phi\sin\theta\,\hat{\mathbf{y}}] = r\sin\theta\,\widehat{\boldsymbol{\phi}}. \tag{25}$$

So in Tensor calculus, the velocity vector $\mathbf{v}$ can be represented as the following function by substitution of Equations 23 to 25 into Equation 22:

$$\mathbf{v} = \frac{dr}{dt}\mathbf{r_r} + \frac{d\theta}{dt}\mathbf{r}_{\boldsymbol{\theta}} + \frac{d\phi}{dt}\mathbf{r}_{\boldsymbol{\phi}}. \tag{26}$$

In terms of unit vectors $\hat{\mathbf{r}}$, $\widehat{\boldsymbol{\theta}}$, and $\widehat{\boldsymbol{\phi}}$, the velocity vector in Equation 26 becomes

$$\mathbf{v} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\widehat{\boldsymbol{\phi}}. \tag{27}$$

Thus, the square of the velocity vector, via vector dot-product, becomes

$$\mathbf{v}\cdot\mathbf{v} = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2. \tag{28}$$

In Tensor calculus, the velocity vector in Equation 26 is defined as a covariant vector that has orthogonal contravariant basis vectors $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$, with radial velocity component defined as $v^r = dr/dt$ and angular velocity components defined as $v^\theta = d\theta/dt$ and $v^\phi = d\theta/dt$, such that the velocity vector is represented instead as

$$\mathbf{v^i} = v^r\mathbf{r_r} + v^\theta\mathbf{r_\theta} + v^\phi\mathbf{r_\phi}, \tag{29}$$

with vector components being the following when using unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$:

$$v^r\mathbf{r_r} = \frac{dr}{dt}\hat{\mathbf{r}} \qquad v^\theta\mathbf{r_\theta} = \frac{d\theta}{dt}r\hat{\boldsymbol{\theta}} \qquad v^\phi\mathbf{r_\phi} = \frac{d\phi}{dt}r\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{30}$$

The superscript **i** used for velocity vector in Equation 29 represents that this velocity vector is a covariant vector in Tensor calculus with the contravariant basis vectors $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$. Because the magnitudes of these basis vectors are not equal one, these three basis vectors are orthogonal but not orthonormal. Since the basis vectors are not orthonormal, in Tensor calculus the square of the magnitude of the velocity vector v by definition is equal to the following vector dot-product instead:

$$v^2 = \mathbf{v^i}\cdot\mathbf{v_i} = \mathbf{v_i}\cdot\mathbf{v^i} = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2. \tag{31}$$

In Equation 31, $\mathbf{v_i}$ is the following contravariant vector with covariant basis vectors $\mathbf{r^r}$, $\mathbf{r^\theta}$, and $\mathbf{r^\phi}$:

$$\mathbf{v_i} = v_r\mathbf{r^r} + v_\theta\mathbf{r^\theta} + v_\phi\mathbf{r^\phi}. \tag{32}$$

From the stipulation in Equation 28, the following vector dot-product applies concerning the contravariant and covariant basis vectors due to the orthogonal condition of the basis vectors:

$$\mathbf{r_i}\cdot\mathbf{r^j} = \delta_i^j = \mathbf{r^j}\cdot\mathbf{r_i} = \delta_j^i = 1 \text{ (If } i = j\text{)}; \tag{33}$$

$$\mathbf{r_i}\cdot\mathbf{r^j} = \delta_i^j = \mathbf{r^j}\cdot\mathbf{r_i} = \delta_j^i = 0 \text{ (If } i \neq j\text{)}. \tag{34}$$

In Equations 33 and 34, i and j represent parameters r, θ, and ϕ and the values of i and j range in value from 1 to 3. In Tensor calculus, x^1 is vector magnitude r, x^2 is angle θ, and x^3 is angle ϕ. Note that i and j are not exponents but indices instead. From the orthogonal conditions given in Equations 33 and 34 and the result from the dot product in Equation 31, covariant basis vectors $\mathbf{r^r}$, $\mathbf{r^\theta}$, and $\mathbf{r^\phi}$ are, therefore, equal to the following functions:

$$\mathbf{r^r} = \cos\phi\sin\theta\,\hat{\mathbf{x}} + \sin\phi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} = \mathbf{r_r} = \hat{\mathbf{r}}; \tag{35}$$

$$\mathbf{r}^{\boldsymbol{\theta}} = \frac{1}{r}\cos\phi\cos\theta\,\hat{\mathbf{x}} + \frac{1}{r}\sin\phi\cos\theta\,\hat{\mathbf{y}} - \frac{1}{r}\sin\theta\,\hat{\mathbf{z}} = \frac{1}{r}\widehat{\boldsymbol{\theta}}; \tag{36}$$

$$\mathbf{r}^{\boldsymbol{\phi}} = -\frac{1}{r\sin\theta}\sin\phi\,\hat{\mathbf{x}} + \frac{1}{r\sin\theta}\cos\phi\,\hat{\mathbf{y}} = \frac{1}{r\sin\theta}\widehat{\boldsymbol{\phi}}. \tag{37}$$

Therefore, vector magnitudes v_r, v_θ, and v_ϕ are defined as follows:

$$v_r = \frac{dr}{dt} \quad \left(v_r\mathbf{r}^{\mathbf{r}} = v^r\mathbf{r}_{\mathbf{r}} = \frac{dr}{dt}\hat{\mathbf{r}}\right); \tag{38}$$

$$v_\theta = r^2\frac{d\theta}{dt} \quad \left(v_\theta\mathbf{r}^{\boldsymbol{\theta}} = v^\theta\mathbf{r}_{\boldsymbol{\theta}} = r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}\right); \tag{39}$$

$$v_\phi = r^2\sin^2\theta\frac{d\phi}{dt} \quad \left(v_\phi\mathbf{r}^{\boldsymbol{\phi}} = v^\phi\mathbf{r}_{\boldsymbol{\phi}} = r\sin\theta\frac{d\phi}{dt}\widehat{\boldsymbol{\phi}}\right). \tag{40}$$

Also, by Tensor calculus, we have the following conditions for the orthogonal basis vectors:

$$\mathbf{r}_\mathbf{i}\cdot\mathbf{r}_\mathbf{j} = g_{ij} = g_{ji} \neq 0 \ \ (\text{If } i = j) \text{ and } \mathbf{r}_\mathbf{i}\cdot\mathbf{r}_\mathbf{j} = g_{ij} = g_{ji} = 0 \ \ (\text{If } i \neq j); \tag{41}$$

$$\mathbf{r}^\mathbf{i}\cdot\mathbf{r}^\mathbf{j} = g^{ij} = g^{ji} \neq 0 \ \ (\text{If } i = j) \text{ and } \mathbf{r}^\mathbf{i}\cdot\mathbf{r}^\mathbf{j} = g^{ij} = g^{ji} = 0 \ \ (\text{If } i \neq j). \tag{42}$$

In spherical polar coordinates, therefore,

$$\mathbf{r}_\mathbf{r}\cdot\mathbf{r}_\mathbf{r} = g_{rr} = 1; \quad \mathbf{r}_{\boldsymbol{\theta}}\cdot\mathbf{r}_{\boldsymbol{\theta}} = g_{\theta\theta} = r^2; \quad \mathbf{r}_{\boldsymbol{\phi}}\cdot\mathbf{r}_{\boldsymbol{\phi}} = g_{\phi\phi} = r^2\sin^2\theta; \tag{43}$$

$$\mathbf{r}^\mathbf{r}\cdot\mathbf{r}^\mathbf{r} = g^{rr} = 1; \quad \mathbf{r}^{\boldsymbol{\theta}}\cdot\mathbf{r}^{\boldsymbol{\theta}} = g^{\theta\theta} = \frac{1}{r^2}; \quad \mathbf{r}^{\boldsymbol{\phi}}\cdot\mathbf{r}^{\boldsymbol{\phi}} = g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}. \tag{44}$$

When taking the first derivative of the velocity vector $\mathbf{v}$ in Equation 26 with respect to time to derive the acceleration vector $\mathbf{a}$, it is important to realize that one has to take the derivative of each vector $\mathbf{r}_\mathbf{r}$, $\mathbf{r}_{\boldsymbol{\theta}}$, and $\mathbf{r}_{\boldsymbol{\phi}}$ with respect to r, θ, and ϕ as well as with respect to time t using the chain rule Therefore, the acceleration vector **a,** the derivative of Equation 26 with respect to time, becomes the following expression:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}_\mathbf{r}\frac{d^2r}{dt^2} + \frac{\partial\mathbf{r}_\mathbf{r}}{\partial r}\left(\frac{dr}{dt}\right)^2 + \frac{\partial\mathbf{r}_\mathbf{r}}{\partial\theta}\frac{dr}{dt}\frac{d\theta}{dt} + \frac{\partial\mathbf{r}_\mathbf{r}}{d\phi}\frac{dr}{dt}\frac{d\phi}{dt} + \mathbf{r}_{\boldsymbol{\theta}}\frac{d^2\theta}{dt^2} + \frac{\partial\mathbf{r}_{\boldsymbol{\theta}}}{\partial r}\frac{dr}{dt}\frac{d\theta}{dt} +$$

$$\frac{\partial\mathbf{r}_{\boldsymbol{\theta}}}{\partial\theta}\left(\frac{d\theta}{dt}\right)^2 + \frac{\partial\mathbf{r}_{\boldsymbol{\theta}}}{\partial\phi}\frac{d\phi}{dt}\frac{d\theta}{dt} + \mathbf{r}_{\boldsymbol{\phi}}\frac{d^2\phi}{dt^2} + \frac{\partial\mathbf{r}_{\boldsymbol{\phi}}}{\partial r}\frac{dr}{dt}\frac{d\phi}{dt} + \frac{\partial\mathbf{r}_{\boldsymbol{\phi}}}{\partial\theta}\frac{d\theta}{dt}\frac{d\phi}{dt} + \frac{\partial\mathbf{r}_{\boldsymbol{\phi}}}{\partial\phi}\left(\frac{d\phi}{dt}\right)^2. \tag{45}$$

In Tensor calculus, derivatives of each vector $\mathbf{r}_\mathbf{r}$, $\mathbf{r}_{\boldsymbol{\theta}}$, and $\mathbf{r}_{\boldsymbol{\phi}}$ with respect to r, θ, and ϕ are defined by the following two expressions:

$$\frac{\partial \mathbf{r_i}}{\partial x^j} = \Gamma_{i,ij}\mathbf{r^i} + \Gamma_{j,ij}\mathbf{r^j} + \Gamma_{k,ij}\mathbf{r^k}; \quad (46)$$

$$\frac{\partial \mathbf{r_i}}{\partial x^j} = \Gamma^i_{ij}\mathbf{r_i} + \Gamma^j_{ij}\mathbf{r_j} + \Gamma^k_{ij}\mathbf{r_k}. \quad (47)$$

In Equations 46 and 47, x^j represents parameters r, θ, and ϕ, and orthogonal contravariant-basis vector $\mathbf{r_i}$, $\mathbf{r_j}$, and $\mathbf{r_k}$ represent vectors $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$ and likewise the same for covariant-basis vectors $\mathbf{r^i}$, $\mathbf{r^j}$, and $\mathbf{r^k}$ represent vectors $\mathbf{r^r}$, $\mathbf{r^\theta}$, and $\mathbf{r^\phi}$. In Tensor calculus, as stated previously, x^1 is vector magnitude r, x^2 is angle θ, and x^3 is angle ϕ. In Equations 46 and 47, $\Gamma_{i,ij}$ is call the Christoffel symbol of the first-kind and Γ^i_{ij} is called the Christoffel symbol of the second-kind, where i, j, and k range from values 1 to 3. With reference to Equations 46 and 47, the Christoffel symbols of the first- and second-kind are the following dot-products due to the orthogonality of the basis vectors:

$$\Gamma_{k,ij} = \frac{\partial \mathbf{r_i}}{\partial x^j}\cdot \mathbf{r_k} \qquad \left[\Gamma_{j,ik} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{x^j}\right) = g_{jj}\Gamma^j_{ik} \text{ since } g_{ij} = 0 \text{ if } i \neq j\right]; \quad (48)$$

$$\Gamma^k_{ij} = \frac{\partial \mathbf{r_i}}{\partial x^j}\cdot \mathbf{r^k} \qquad \left(\Gamma^j_{ik} = \sum_{l=1}^{3} g^{jl}\Gamma_{l,ik} = g^{jj}\Gamma_{j,ik} \text{ since } g^{ij} = 0 \text{ if } i \neq j\right). \quad (49)$$

What is also shown in each parenthesis for both Equations 48 and 49 is another way to evaluate both the first- and second-kind of Christoffel symbols.

When using mathematics to quantitatively analyze gravity from Einstein's General Theory of Relativity, the Christoffel symbol of the second-kind is use instead of the first-kind. Since there are 3 dimensions in spherical polar coordinates, there are a total of $3^3 = 27$ different Christoffel symbols of the first- and second-kind each, but only 9 out of the 27 of the first- and second-kind turn out not to be equal to zero using Equations 48 and 49. The 9 Christoffel symbols of the first- and second-kind not equal to zero are shown in Table I, and 3 pairs of the 9 non-zero Christoffel symbols of the first- and second-kind being equal because of the condition $\Gamma_{k,ij} = \Gamma_{k,ji}$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$.

If one substitutes the summations in Equation 46 for the derivatives $d\mathbf{r}_i/x^j$ into Equation 45 for the acceleration vector **a,** collect all like terms for basis vectors $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$, one has for each component, $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$, the following expression of the acceleration vector **a**:

$$\mathbf{a}_i = \left(\frac{d^2x^i}{dt^2} + \sum_{j=1}^{3}\sum_{k=1}^{3}\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)\mathbf{r_i}. \quad (50)$$

Table I. Christoffel Symbols for spherical polar coordinates.

Christoffel Symbols of the Second-Kind	Christoffel Symbols of the First-Kind
$\Gamma^r_{\theta\theta} = \mathbf{r^r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \Gamma^1_{22} = -r$	$\Gamma_{r,\theta\theta} = \mathbf{r_r} \cdot \frac{\partial \mathbf{r_\theta}}{\partial \theta} = -r$
$\Gamma^r_{\phi\phi} = \mathbf{r^r} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \phi} = \Gamma^1_{33} = -r\sin^2\theta$	$\Gamma_{r,\phi\phi} = \mathbf{r_r} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \phi} = -r\sin^2\theta$
$\Gamma^\theta_{r\theta} = \mathbf{r^\theta} \cdot \frac{\partial \mathbf{r_r}}{\partial \theta} = \Gamma^\theta_{\theta r} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$	$\Gamma_{\theta,r\theta} = \mathbf{r_\theta} \cdot \frac{\partial \mathbf{r_r}}{\partial \theta} = \Gamma_{\theta,\theta r} = r$
$\Gamma^\theta_{\phi\phi} = \mathbf{r^\theta} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \phi} = \Gamma^2_{33} = -\cos\theta\sin\theta$	$\Gamma_{\theta,\phi\phi} = \mathbf{r_\theta} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \phi} = -r^2\cos\theta\sin\theta$
$\Gamma^\phi_{r\phi} = \mathbf{r^\phi} \cdot \frac{\partial \mathbf{r_r}}{\partial \phi} = \Gamma^\phi_{\phi r} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$	$\Gamma_{\phi,r\phi} = \mathbf{r_\phi} \cdot \frac{\partial \mathbf{r_r}}{\partial \phi} = \Gamma_{\phi,\phi r} = r\sin^2\theta$
$\Gamma^\phi_{\theta\phi} = \mathbf{r^\phi} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \theta} = \Gamma^\phi_{\phi\theta} = \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$	$\Gamma_{\phi,\phi\theta} = \mathbf{r_\phi} \cdot \frac{\partial \mathbf{r_\phi}}{\partial \theta} = \Gamma_{\phi,\theta\phi} = r^2\sin\theta\cos\theta$

Also, when one substitutes only those Christoffel symbols of the second-kind which are not equal to zero, Equation 45 becomes the following expression:

$$\frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]\mathbf{r_r} + \left[\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} - \cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2\right]\mathbf{r_\theta} + \left[\frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} + 2\cot\theta\frac{d\theta}{dt}\frac{d\phi}{dt}\right]\mathbf{r_\phi}. \quad (51)$$

With constant speed for tangential velocity along a curve of a surface, meaning no change in kinetic energy and shortest distance between two points on a curve, each of the magnitudes within the brackets, for all three components $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$, will be equal to zero in Equation 51. This is a result for the traditional Geodesic equation being equal to zero concerning the minimum distance between two points on a curved path:

$$\frac{d^2x^i}{dt^2} + \sum_{j=1}^{3}\sum_{k=1}^{3}\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \quad (52)$$

With regard to each term in the expression of Equation 52, zero acceleration implies the following equations being equal to zero:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 = 0; \quad (53)$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} - \cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2 = 0; \quad (54)$$

$$\frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} + 2\cot\theta\frac{d\theta}{dt}\frac{d\phi}{dt} = 0. \quad (55)$$

If the angle θ is equal to π/2 radians (or for a 90° angle) and is constant, Equations 53 to 55 become:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0; \quad (56)$$

$$\frac{d^2\theta}{dt^2} = 0; \quad (57)$$

$$\frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} = 0. \quad (58)$$

Equation 56 matches with circular motion where radial acceleration is zero, such as a perfect circular planetary orbit, and the $r(d\theta/dt)^2$accounts for centripetal acceleration towards the center. In addition, when multiplying r through the expression given in Equation 58, the second term in Equation 58 represents the Coriolis force $2(dr/dt)(d\phi/dt)$, which one feels as they walk outward or inward in the radial direction of a rotating Carousel or Merry-Go-Round.

The derivation of the Geodesic expression in Equation 52 being equal to zero is obtained from the following Lagrangian L for motion of constant speed v in three-dimensions first by multiplying ½ though Equation 28:

$$L = \frac{1}{2}v^2 = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2. \quad (59)$$

If one has constant speed moving between two points, and for minimum distance between two points on a curve, the following expression, known as the Euler-Lagrange equation (see Fox, 1963), is equal to zero for all three components of $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(dx^i/dt)}\right) - \frac{\partial L}{\partial x^i} = 0. \quad (60)$$

Applying the condition in Equation 60 for terms in Equation 59 will also result in determining which Christoffel symbols of the second-kind are not equal to zero in spherical polar coordinates. When the expression in Equation 60 is set equal to zero, this yields the minimum distance between two points on a curved path which also matches up with constant speed for motion upon a curved surface.

Application of Tensor Calculus in Einstein's General Theory of Relativity

For Minkowski space, Equation 8, represented in spherical polar coordinates becomes the following expression with reference to Equation 28:

$$-c^2 d\tau^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \quad (61)$$

Theoretically, gravitational force is due to the stretching of Minkowski space, or four-dimensional space-time. When Schwarzschild developed his metric using Einstein's General Theory of Relativity, he assumed the stretching was about time t and radial distance r for a spherically shaped planet, star, collapsed star, or black hole in space. His initial expression for stretched Minkowski space is the following function involving exponents:

$$-c^2 d\tau^2 = -e^a c^2 dt^2 + e^{-a} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \quad (62)$$

Karl Schwarzschild initially set both exponents as the same two unknown functions of radial distance r, $a = a(r)$, with one being positive in the first addition term in Equation 62 and negative in the second addition term. He made this choice in order that the Minkowski space could be stretched but not curved. This correlates with the theory that outside a planet, star, collapsed star, or black hole, Minkowski space should not be curved but only stretched. If we divide both sides of Equation 62 with $d\tau^2$ and multiply through by ½, one has the following Lagrangian L:

$$L = -\frac{1}{2}c^2 = -\frac{1}{2}e^a c^2 \left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}e^{-a}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{d\tau}\right)^2 + \frac{1}{2}r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2. \quad (63)$$

Unknown function of r, $a = a(r)$, should be evaluated using the mandatory condition that the diagonal elements of the Ricci tensor $\mathcal{R}_{ii}$ are all equal to zero. This is true if there is no curvature in the four-dimensions of Minkowski space.

All the off-diagonal elements $\mathcal{R}_{ij}$ are, of course, equal to zero due to the orthogonality of unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{t}}$. There is a total of 16 elements (4×4) in the Ricci tensor which is a 4-by-4 matrix. When all of these 16 elements in the Ricci tensor are equal to zero, then there is no curvature in the four-dimensional space-time outside the planet, star, collapsed star, or black hole. But first, we must use the Euler-Lagrange equation to evaluate which Christoffel symbols of the second-kind in Minkowski space that are not equal to zero, by using the mathematical condition in Minkowski space that the speed is always equal to c, that of light, because there is no acceleration or deceleration. This is similar to the derivation of Equation 51 for curved motion in three-dimensions parameterized by time that all the components for basis vectors $\mathbf{r_r}$, $\mathbf{r_\theta}$, and $\mathbf{r_\phi}$ are equal to zero. We first determine the Geodesic equation for time t coordinate x^0, such that for the four-dimensions indices 0 is used for the time axis, and as previously 1 for r, 2 for θ, and 3 for ϕ. We take the first derivative of Equation 63 with respect to proper time τ:

$$\frac{\partial L}{\partial \tau} = 0, \quad (64)$$

and then we take the first derivative of Equation 63 with respect to $dt/d\tau$:

$$\frac{\partial L}{\partial(dt/d\tau)} = -c^2 e^a \frac{dt}{d\tau}. \qquad (65)$$

The next step is to take the derivative of Equation 65 with respect to proper time τ:

$$\frac{d}{d\tau}\left[\frac{\partial L}{\partial(dt/d\tau)}\right] = -c^2 \frac{da}{dr} e^a \frac{dr}{d\tau}\frac{dt}{d\tau} - c^2 e^a \frac{d^2t}{d\tau^2}. \qquad (66)$$

Afterwards, one substitutes Equations 65 and 66 into Equation 60, the Euler-Lagrange equation becomes the following expression:

$$\frac{d}{d\tau}\left[\frac{\partial L}{\partial(dt/d\tau)}\right] - \frac{\partial L}{\partial t} = -c^2 \frac{da}{dr} e^a \frac{dr}{d\tau}\frac{dt}{d\tau} - c^2 e^a \frac{d^2t}{d\tau^2} - 0 = 0. \qquad (67)$$

To obtain the Geodesic equation, one simply multiplies Equation 67 with negative one and then divide by the product $c^2 e^a$:

$$\frac{d^2t}{d\tau^2} + \frac{da}{dr}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0. \qquad (68)$$

In comparison with the Geodesic equation in Equation 52, replacing time t with proper time τ, Equation 52 is the following expression:

$$\frac{d^2x^i}{d\tau^2} + \sum_{j=0}^{n}\sum_{k=0}^{n} \Gamma^i_{jk} \frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0. \qquad (69)$$

When comparing Equation 69 with Equation 68, obviously there are only two Christoffel symbols of the second-kind that are not equal to zero and are also equal to one another. Because of the double summation in the Geodesic equation for coordinates t and r (x^0 and x^1 for $j = 0$, $k = 1$ and $j = 1$, $k = 0$), the Christoffel symbol is one-half of the first-derivative of function a with respect to r, da/dr. Table II lists all the Christoffel symbols of the second-kind derived using the Euler-Lagrange equation for Equation 63 involving the function $a(r)$ to be determined. There are a total of 14 Christoffel symbols of the second-kind that are not equal to zero when the other 50 are equal to zero. There are a total of 64 Christoffel symbols of the second-kind due to having four-dimensions, 4×4×4 = 64.

To evaluate the exponent a as some function of r, we will need to set the diagonal terms of the Ricci tensor to zero. We have up to 16 contravariant Riemann curvature tensor values $\mathcal{R}^i_{kik}$ to calculate, four for each Ricci tensor components $\mathcal{R}_{ii}$, with i and k equal 0 to 3 for all four dimensions. By definition (see Cheng, 2010), the covariant and contravariant Riemann tensors are the following two expressions for 4 dimensions of Minkowski space:

$$\mathcal{R}_{kmji} = \frac{\partial \Gamma_{k,mi}}{\partial x^j} - \frac{\partial \Gamma_{k,mj}}{\partial x^i} + \sum_{n=0}^{3} \Gamma_{n,mj}\Gamma_{ki}^{n} - \sum_{n=0}^{3} \Gamma_{n,mi}\Gamma_{kj}^{n}\,; \tag{70}$$

$$\mathcal{R}_{mji}^{k} = \frac{\partial \Gamma_{mi}^{k}}{\partial x^j} - \frac{\partial \Gamma_{mj}^{k}}{\partial x^i} + \sum_{n=0}^{3} \Gamma_{nj}^{k}\Gamma_{mi}^{n} - \sum_{n=0}^{3} \Gamma_{ni}^{k}\Gamma_{mj}^{n}. \tag{71}$$

Table II Christoffel symbols of the second-kind before and after deriving the Schwarzschild metric with $\boldsymbol{a(r) = \log_e[1 - GM_0/(c^2 r)]}$

$\Gamma_{rt}^{t} = \Gamma_{tr}^{t} = \Gamma_{10}^{0} = \Gamma_{01}^{0} = \frac{1}{2}\frac{da}{dr}$	$\Gamma_{rt}^{t} = \Gamma_{tr}^{t} = \Gamma_{10}^{0} = \Gamma_{01}^{0} = \frac{GM_0/(c^2r^2)}{1 - 2GM_0/(c^2 r)}$
$\Gamma_{tt}^{r} = \Gamma_{00}^{1} = \frac{1}{2}\frac{da}{dr}e^{2a}c^2$	$\Gamma_{tt}^{r} = \Gamma_{00}^{1} = \frac{GM_0}{r^2}\left(1 - \frac{2GM_0}{c^2 r}\right)$
$\Gamma_{rr}^{r} = \Gamma_{11}^{1} = -\frac{1}{2}\frac{da}{dr}$	$\Gamma_{rr}^{r} = \Gamma_{11}^{1} = -\frac{GM_0/(c^2r^2)}{1 - 2GM_0/(c^2 r)}$
$\Gamma_{\theta\theta}^{r} = \Gamma_{22}^{1} = -e^{a}r$	$\Gamma_{\theta\theta}^{r} = \Gamma_{22}^{1} = -\left(1 - \frac{2GM_0}{c^2 r}\right)r$
$\Gamma_{\phi\phi}^{r} = \Gamma_{33}^{1} = -e^{a}r\sin^2\theta$	$\Gamma_{\phi\phi}^{r} = \Gamma_{33}^{1} = -\left(1 - \frac{2GM_0}{c^2 r}\right)r\sin^2\theta$
$\Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}$	
$\Gamma_{\phi\phi}^{\theta} = \Gamma_{33}^{2} = -\sin\theta\cos\theta$	
$\Gamma_{r\phi}^{\phi} = \Gamma_{\phi r}^{\phi} = \Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r}$	
$\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \Gamma_{23}^{3} = \Gamma_{32}^{3} = \cot\theta$	

In flat space, no matter the number of dimensions, the two expressions in Equations 70 and 71 will all be equal to zero, but not in curved or stretched space. The two expressions in Equations 70 and 71 are derived from the following double derivative when using Tensor calculus, which we are showing an example of for 3 dimensions in spherical polar coordinates:

$$D_r D_\theta \mathbf{R}(r,\theta,\phi) - D_\theta D_r \mathbf{R}(r,\theta,\phi) = f(r,\theta,\phi). \tag{72}$$

$D_r\mathbf{R}(r,\theta,\phi)$ is the following total derivative of a function:

$$D_r\mathbf{R}(r,\theta,\phi) = \frac{\partial}{\partial r}\left[R_r(r,\theta,\phi)\mathbf{r}_\mathrm{r} + R_\theta(r,\theta,\phi)\mathbf{r}_\theta + R_\phi(r,\theta,\phi)\mathbf{r}_\phi\right]. \tag{73}$$

Note that components of the **R** vector are all functions of the three spherical polar coordinates. Thus, just taking the first derivate with respect to r yields the following:

$$D_r\mathbf{R}(r,\theta,\phi) = \frac{\partial R_r}{\partial r}\mathbf{r_r} + R_r\frac{\partial \mathbf{r_r}}{\partial r} + \frac{\partial R_\theta}{\partial r}\mathbf{r_\theta} + R_\theta\frac{\partial \mathbf{r_\theta}}{\partial r} + \frac{\partial R_\phi}{\partial r}\mathbf{r_\phi} + R_\phi\frac{\partial \mathbf{r_\phi}}{\partial r}. \quad (74)$$

Note that this involves taking partial derivatives of the basis vectors which also result in further expansions using the Christoffel symbols of either the first- or second-kind. Using the first-kind gives the result in Equation 70, and use of the second-kind yields the result in Equation 71. This mathematics is very tedious and lengthy, and so it will not be shown in further detail. However, if one works out the math for the subtraction of the double derivative function $D_rD_\theta\mathbf{R}(r,\theta,\phi) - D_\theta D_r\mathbf{R}(r,\theta,\phi)$ and collects like terms, one has either of the two forms of the Riemann curvature tensor for basis vector $\mathbf{r^k}$ or $\mathbf{r_k}$:

$$\mathcal{R}_{kmji}\mathbf{r^k} = \left(\frac{\partial\Gamma_{k,mi}}{\partial x^j} - \frac{\partial\Gamma_{k,mj}}{\partial x^i} + \sum_{n=0}^{3}\Gamma_{n,mj}\Gamma^n_{ki} - \sum_{n=0}^{3}\Gamma_{n,mi}\Gamma^n_{kj}\right)\mathbf{r^k}; \quad (75)$$

$$\mathcal{R}^k_{mji}\mathbf{r_k} = \left(\frac{\partial\Gamma^k_{mi}}{\partial x^j} - \frac{\partial\Gamma^k_{mj}}{\partial x^i} + \sum_{n=0}^{3}\Gamma^k_{nj}\Gamma^n_{mi} - \sum_{n=0}^{3}\Gamma^k_{ni}\Gamma^n_{mj}\right)\mathbf{r_k}. \quad (76)$$

Note that if $D_rD_\theta\mathbf{R}(r,\theta,\phi) - D_\theta D_r\mathbf{R}(r,\theta,\phi) = f(r,\theta,\phi) = 0$, both terms in the parenthesis above will be equal to zero for all basis vectors $\mathbf{r^k}$ and $\mathbf{r_k}$ due to the space being flat and not stretched.

The Riemann tensor matrix has up to 256 curvature components due to the presence of four dimensions (4×4×4×4=256). By another definition (see Cheng, 2010), the Ricci curvature tensor components $\mathcal{R}_{ii}$ are equal to the following summation of the contravariant Riemann curvature tensor components for the four dimensions in Minkowski space:

$$\mathcal{R}_{ii} = \sum_{j=0}^{3}\mathcal{R}^j_{iji}. \quad (77)$$

To calculate $\mathcal{R}_{00}$ or $\mathcal{R}_{tt}$, we must first calculate the following Riemann curvature tensor components $\mathcal{R}^0_{000}$, $\mathcal{R}^1_{010}$, $\mathcal{R}^2_{020}$, and $\mathcal{R}^3_{030}$ as follows. The first Riemann curvature tensor component $\mathcal{R}^0_{000}$ or $\mathcal{R}^t_{ttt}$ is simply equal to zero since the terms being subtracted out are identical:

$$\mathcal{R}^0_{000} = \frac{\partial\Gamma^0_{00}}{\partial x^0} - \frac{\partial\Gamma^0_{00}}{\partial x^0} + \sum_{n=0}^{3}\Gamma^0_{n0}\Gamma^n_{00} - \sum_{n=0}^{3}\Gamma^0_{n0}\Gamma^n_{00} \quad (\mathcal{R}^t_{ttt} = 0). \quad (78)$$

However, we will need to evaluate $\mathcal{R}^1_{010}$ or $\mathcal{R}^r_{trt}$:

$$\mathcal{R}^1_{010} = \frac{\partial \Gamma^1_{00}}{\partial x^1} - \frac{\partial \Gamma^1_{01}}{\partial x^0} + \sum_{n=0}^{3} \Gamma^1_{n1}\Gamma^n_{00} - \sum_{n=0}^{3} \Gamma^1_{n0}\Gamma^n_{01}. \quad (79)$$

Because the three Christoffel symbols of the second-kind Γ^0_{01}, Γ^1_{00}, and Γ^1_{11} or Γ^t_{tr}, Γ^r_{tt}, and Γ^r_{rr} are not equal to zero:

$$\mathcal{R}^1_{010} = \frac{\partial \Gamma^1_{00}}{\partial x^1} + \Gamma^1_{11}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{01} \quad \left(\mathcal{R}^r_{trt} = \frac{\partial \Gamma^r_{tt}}{\partial r} + \Gamma^r_{rr}\Gamma^r_{tt} - \Gamma^r_{tt}\Gamma^t_{tr}\right). \quad (80)$$

After substituting in Γ^0_{01}, Γ^1_{00}, and Γ^1_{11} given in Table II before deriving the Schwarzschild metric, Equation 80 becomes the following expression:

$$\mathcal{R}^1_{010} = \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right) + \left(-\frac{1}{2}\frac{da}{dr}\right)\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right) - \left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right)\left(\frac{1}{2}\frac{da}{dr}\right); \quad (81)$$

$$\mathcal{R}^1_{010} = \frac{1}{2}\frac{d^2a}{dr^2}e^{2a}c^2 + \frac{1}{2}\left(\frac{da}{dr}\right)^2 e^{2a}c^2 = \frac{1}{2}\left[\frac{d^2a}{dr^2} + \left(\frac{da}{dr}\right)^2\right]e^{2a}c^2. \quad (82)$$

Since two of the following Christoffel symbols of the second-kind Γ^1_{00} and Γ^2_{12} or Γ^r_{tt} and $\Gamma^\theta_{r\theta}$ are not equal to zero, likewise we need to evaluate Riemann curvature tensor component $\mathcal{R}^2_{020}$ or $\mathcal{R}^\theta_{t\theta t}$:

$$\mathcal{R}^2_{020} = \Gamma^2_{12}\Gamma^1_{00} \quad \left(\mathcal{R}^\theta_{t\theta t} = \Gamma^\theta_{r\theta}\Gamma^r_{tt}\right); \quad (83)$$

$$\mathcal{R}^2_{020} = \frac{1}{r}\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right). \quad (84)$$

Concerning the Riemann curvature tensor component $\mathcal{R}^3_{030}$ or $\mathcal{R}^\phi_{t\phi t}$, the Christoffel symbols of the second-kind Γ^1_{00} and Γ^3_{13} or Γ^r_{tt} and $\Gamma^\phi_{r\phi}$ are not equal to zero, so the computation of $\mathcal{R}^3_{030}$ is as follows:

$$\mathcal{R}^3_{030} = \Gamma^3_{13}\Gamma^1_{00} \quad \left(\mathcal{R}^\phi_{t\phi t} = \Gamma^\phi_{r\phi}\Gamma^r_{tt}\right); \quad (85)$$

$$\mathcal{R}^3_{030} = \frac{1}{r}\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right). \quad (86)$$

So, our first diagonal element of the Ricci curvature tensor $\mathcal{R}_{00}$ is the following summation:

$$\mathcal{R}_{00} = \mathcal{R}^0_{000} + \mathcal{R}^1_{010} + \mathcal{R}^2_{020} + \mathcal{R}^3_{030}; \quad (87)$$

$$\mathcal{R}_{00} = 0 + \frac{1}{2}\frac{d^2a}{dr^2}e^{2a}c^2 + \frac{1}{2}\left(\frac{da}{dr}\right)^2 e^{2a}c^2 + \frac{1}{r}\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right) + \frac{1}{r}\left(\frac{1}{2}\frac{da}{dr}e^{2a}c^2\right). \quad (88)$$

Factoring out $e^{2a}c^2$ and $1/2$ we obtain the following expression for Equation 88:

$$\mathcal{R}_{00} = \frac{1}{2}\left[\frac{d^2a}{dr^2} + \left(\frac{da}{dr}\right)^2 + \frac{2}{r}\frac{da}{dr}\right]e^{2a}c^2. \quad (89)$$

For $\mathcal{R}_{00}$ to be equal to zero, the expression, within the brackets, of Equation 89 must be equal to zero:

$$\frac{d^2a}{dr^2} + \left(\frac{da}{dr}\right)^2 + \frac{2}{r}\frac{da}{dr} = 0. \quad (90)$$

Table III displays all the diagonal elements of the Ricci curvature tensor calculated using the above discussion. In Table III, it is important to note that the Ricci curvature tensor $\mathcal{R}_{11}$ contains the expression of Equation 90 within brackets also. So if Equation 90 is equal to zero, $\mathcal{R}_{11}$ likewise is equal to zero.

The next step is by setting equal to zero the Ricci curvature tensor $\mathcal{R}_{22}$ (Table III) ensuring that Ricci curvature tensor $\mathcal{R}_{33}$ is also equal to zero:

$$-\left(r\frac{da}{dr} + 1\right)e^a + 1 = 0. \quad (91)$$

Table III **The Ricci curvature tensors $\mathcal{R}_{ii}$ for $i = 0$ to 3 (t, r, θ, ϕ) set equal to zero to derive the Schwarzschild metric determining that $a(r) = \log_e[1 - GM_0/(c^2r)]$**

$$\mathcal{R}_{00} = \frac{1}{2}\left[\frac{d^2a}{dr^2} + \left(\frac{da}{dr}\right)^2 + \frac{2}{r}\frac{da}{dr}\right]e^{2a}c^2$$

$$\mathcal{R}_{11} = -\frac{1}{2}\left[\frac{d^2a}{dr^2} + \left(\frac{da}{dr}\right)^2 + 2\frac{da}{dr}\frac{1}{r}\right]$$

$$\mathcal{R}_{22} = -\left(\frac{da}{dr}r + 1\right)e^a + 1$$

$$\mathcal{R}_{33} = \left[-\left(\frac{da}{dr}r + 1\right)e^a + 1\right]\sin^2\theta = \mathcal{R}_{22}\sin^2\theta$$

To evaluate the exponent a as a function of radius r, we will use the following static situation:

$$dr = 0, \qquad d\theta = 0, \qquad d\phi = 0. \tag{92}$$

Insert the Christoffel symbols of the second-kind into the Geodesic equation for coordinate r (Table II):

$$\frac{d^2r}{d\tau^2} + \frac{1}{2}\frac{da}{dr}e^{2a}c^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}\frac{da}{dr}\left(\frac{dr}{d\tau}\right)^2 - e^a r\left(\frac{d\theta}{d\tau}\right)^2 - e^a r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \tag{93}$$

and Equation 93 becomes the following expression for the static condition:

$$\frac{d^2r}{d\tau^2} + \frac{1}{2}\frac{da}{dr}e^{2a}c^2\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{94}$$

In theory, Equation 94 should match Newtonian gravity because we are using the static condition:

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{da}{dr}e^{2a}c^2\left(\frac{dt}{d\tau}\right)^2 = -\frac{GM_0}{r^2}. \tag{95}$$

In Equation 95, M_0 is the rest mass of the sun and G is the constant for Newton's Universal Law of Gravitation. To determine the value of the squared differential expression $(dt/d\tau)^2$, we will use the same static case, dividing $d\tau^2$ through Equation 62 for the Schwarzschild metric, or stretched Minkowski space, in the following expression:

$$-c^2 = -c^2e^a\left(\frac{dt}{d\tau}\right)^2 + e^{-a}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2, \tag{96}$$

and because of the static case in Equation 96, $(dt/d\tau)^2$ is equal to e^{-a} or $(dt/d\tau)^2 = e^{-a}$. Thus, we can now substitute e^{-a} for $(dt/d\tau)^2$ in Equation 95 to obtain:

$$\frac{1}{2}\frac{da}{dr}e^{2a}c^2\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2}\frac{da}{dr}e^a c^2 = \frac{GM_0}{r^2}. \tag{97}$$

Therefore,

$$\frac{da}{dr} = \left(\frac{2GM_0}{c^2r^2}\right)e^{-a}. \tag{98}$$

One can then substitute the expression of Equation 98 into Equation Ricci tensor component $\mathcal{R}_{22} = 0$ from Table III for da/dr to arrive at:

$$-\left[\left(\frac{2GM_0}{c^2r^2}\right)e^{-a}r + 1\right]e^a + 1 = 0, \tag{99}$$

and if one solves for unknown a as a function of radial distance r using Equation 99, the result is the following natural logarithmic function:

$$a = \log_e \left(1 - \frac{2GM_0}{c^2 r}\right). \tag{100}$$

Thus, the Schwarzschild metric becomes

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM_0}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM_0}{c^2 r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{101}$$

From Equation 101, the radius r of a black hole in outer spaced produced by the collapse of a sufficient large star is equal to the ratio $2GM_0/c^2$.

The final step is to utilize Equation 100 in Equation 90 to see if $\mathcal{R}_{00} = 0$ and $\mathcal{R}_{11} = 0$ (See Table III). Initially, take the first- and second-derivatives of Equation 100 with respect to r, and then square the first-derivative. Substituting these into Equation 90, the term in the brackets of Ricci curvature tensors $\mathcal{R}_{00}$ and $\mathcal{R}_{11}$, one will note that Equation 90 is equal to zero matching with $\mathcal{R}_{11}$ and $\mathcal{R}_{00}$ being zero in value:

$$\frac{d^2 a}{dr^2} + \left(\frac{da}{dr}\right)^2 + 2\frac{da}{dr}\frac{1}{r} = -\frac{\left(\frac{2GM_0}{c^2 r^2}\right)^2}{\left(1 - \frac{2GM_0}{c^2 r}\right)^2} - 2\left(\frac{\frac{2GM_0}{c^2 r^2}}{1 - \frac{2GM_0}{c^2 r}}\right)\frac{1}{r} + \frac{\left(\frac{2GM_0}{c^2 r^2}\right)^2}{\left(1 - \frac{2GM_0}{c^2 r}\right)^2} + 2\left(\frac{\frac{2GM_0}{c^2 r^2}}{1 - \frac{2GM_0}{c^2 r}}\right)\frac{1}{r} = 0. \tag{102}$$

The first and third terms of Equation 102 add up to zero, and the second and fourth terms likewise add up to zero also, because of the way Schwarzschild had assumed the exponents in Equation 62 would yield $\mathcal{R}_{00}$ and $\mathcal{R}_{11}$ both being equal to zero. Table II lists the Christoffel symbols of the second-kind that match up the Schwarzschild metric as shown in Equation 101.

Conclusion

If a collapsing star had sufficient mass to form a black hole instead of a white dwarf, it would take infinity for the collapsing star to reach the radius of a black hole, as calculated for a black hole using the Schwarzschild metric $2GM_0/c^2$, because of time dilation to a value of zero at this radius value. If one could ever vision an object falling into a black hole, initially they would see it accelerate in velocity, but due to time dilation in very strong gravitational fields, as the falling object approaches the surface of a black hole, or even a white dwarf, they would see the object begin to slow down in speed due to the time dilation effects of strong gravitational fields. This is one of the mysteries of our universe, and we humans may find it a challenge to observe experimentally, due to our short life-time expectancies and the vast distances of our universe.

Finally, if one performs the correct number of algebraic manipulations on Equation 101, the Schwarzschild metric, the following equation results:

$$\left[\left(1-\frac{2GM}{c^2r}\right)^2\left(\frac{dt}{d\tau}\right)^2-1\right]\frac{1}{2}m_0c^2=\frac{1}{2}m_0v^2-\frac{GM_0m_0}{r}-\frac{GM_0m_0}{c^2}r\left(\frac{d\phi}{d\tau}\right)^2. \tag{103}$$

Note that the expression in Equation 103 matches the classical Newtonian gravity for planetary motion around the sun for the first two terms on the right-hand-side of the equation, the summation of the kinetic energy and the negative gravitational potential energy:

$$\frac{1}{2}m_0v^2-\frac{GM_0m_0}{r}=E \quad (E<0 \text{ for bound orbits}) \quad (E\geq 0 \text{ for unbound orbits}). \tag{104}$$

The third term becomes important if for a stable orbit, the planetary speed begins to approach that of light in a very strong gravitational field such as that of a white dwarf or black hole. For orbital speeds much less than that of light, the third term in Equation 103 is nearly equal to zero:

$$\frac{GM_0m_0}{c^2}r\left(\frac{d\phi}{d\tau}\right)^2\approx 0 \text{ for speeds much less than that of light } (v\ll c).$$

Therefore, Equation 103 approaches the expression in Equation 104 for orbital speeds much less than that of light.

Karl Schwarzschild is a much-underappreciated physicist of the 20th century. He was an experimental and theoretical physicist. He developed tools and concepts that are still salient in astronomy. He came up with the concept of spectral type and the color of a star, and developed tools with course grating that measured the separation of stars. Schwarzschild also made significant contributions to quantum theory by explaining the Stark Effect or how light splits in an electric field. Schwarzschild died too early at the age of 43 in 1916 from complications of war injuries. He never had a chance to improve upon his 1915 paper, as Droste and Hilbert did. Schwarzschild, with his short life, made a huge impact on physics, and it is hard to name another 20th century physicist that impacted quantum theory and relativity and astronomy, with the theory and the experiments. It is equally important to remember other key investigators such as Droste and Hilbert who elaborated further on our understanding of Einstein's Theory of Relativity. For example, Droste's work on Repulsive Gravity is considered by some researchers to be a possible basis of Dark Energy (Droste, 1916).

In this paper, we have undertaken a unique approach to explain Einstein's Theory of Relativity for the educated mathematician, and provided some historical aspects on how these ideas evolved. We have demonstrated the effect on light and the origin of black holes by using the first non-trivial solution to the Einstein Field Equations. The further exploitation of Einstein's Field Equations led by other scientists such as Stephen Hawking has been more recent and much of it still trails back to these initial contributions on black holes. Much of modern physics owes a great debt to these early investigators.

As a society, we are fortunate that Einstein's General Theory of Relativity inspired many talented investigators such as Schwarzschild, Droste, and Hilbert. The practical applications from concepts of these investigators have been immense on society. For instance, global navigation satellite systems are more accurate with use of relativistic corrections, which has many implications in agriculture, communications, conservation, real estate, meteorology, military, travel, and other vocations where navigation precision is important. In a nutshell, Einstein's Theory of Relativity explains why different observers, traveling at different speeds can and will have different perspectives about their surroundings. We do not know where this concept will take humanity even in the next few decades. It is understood that a person or object that exceeds the speed of light may, indeed, travel into the future, but is that really possible? It is not much of a prognostication that we will call 21st century physics the *Age of Relativity.*

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